

Fatigue Life Prediction of Aluminum Sheets Using Different Probabilistic Methods

Tariq R. Aboalhol¹, Khaled B. Abdusamad², Ali A. Shommaki¹,
Yasin A. Amer¹

¹The Higher Technical Center for Training and Production, ²The Libyan
Academy for Graduate Studies

Corresponding author email: taboalhol@tpc.ly

المخلص

بيانات اختبارات اجهاد الكلال لمكونات الآلات الميكانيكية بها الكثير من التشتت وعدم الاتساق للتنبؤ بانهيائها. بسبب عدم اليقين في التقدير بعمر القطعة تحت إجهادات دورية والمنكررة، تم اقتراح أساليب احتمالية لتحديد مدى تأثير عدم اليقين على معرفة عمر المكون تحت دورات التحميل. محاكاة مونت كارلو هي الطريقة الشائعة الاستخدام لحساب ذلك، لكن كلفة الزمن الحسابي المرتبط بمحاكاة باهظاً أثناء تقييمات مرحلة التصميم. لذلك، كان الغرض من هذه الدراسة هو توضيح استخدام طرق احتمالية أخرى وفعالة، كبداية لطريقة مونت كارلو، للتنبؤ بدقة بعمر عينات من ألواح الألومنيوم T6 – 6061 الخاضعة لإجهادات الكلال. بناءً على توزيعات احصائية لإجهاد الكلال مقابل عمر التحميل المتكرر قبل الانهيار والمحددة معملياً، ظهر اتفاق بين التوزيعات المتوقعة لعمر العينات بشكل جيد جداً مع نتائج الاختبار المعملية. أظهرت الدراسة أنه مع وقت حسابي أقل بشكل كبير، هناك اتفاق وثيق بين كل من طريقة المتقدمة لحساب القيمة المتوسطة (AMV) وطريقة الموثوقية من الدرجة الأولى (FORM) مع محاكاة مونت كارلو. ومع ذلك، كانت طريقة القيمة المتوسطة (MV) أقل دقة في التنبؤ بالقيم المتطرفة عن المتوسط المعياري للعينات والتي تشير إلى ضعف التنبؤ عمر القطعة تحت الاجهاد الكلال. أظهر تحليل حساسية متغيرات، بمعادلة تقدير الاجهاد والعمر على التنبؤات، أن معلمات انحدار (a, b) لمنحنى $S - N$ لها مساهمات ضئيلة التأثير في عمر العينة تحت الكلال مقارنة بتأثير كل من الحمل الديناميكي المتكرر والابعاد الهندسية للعينات.

Abstract

Components under fatigue loading have significant amounts of scatter and inconsistency considered in fatigue failure data. Due to uncertainties in predicting component fatigue life, probabilistic approaches were to quantify the influence of uncertainties on the fatigue life. However, computation time associated with the commonly used Monte Carlo simulation can be prohibitive during design stage assessments. Therefore, the purpose of the current work was to demonstrate the use of efficient probabilistic methods, as alternatives to Monte Carlo method, to accurately predict fatigue lives for Aluminum 6061-T6 sheets. Based on experimentally determined of stress-life distributions, predicted distributions of fatigue life agreed closely with replicate experimental test results. The study showed that with significantly less computational time, there is a very good agreement for advanced mean value (AMV) and the first order reliability method (FORM) with Monte Carlo. However, the mean-value (MV) method was less accurate in tail regions that indicates poor life prediction. The sensitivity analysis showed that S-N slope parameters (a and b) had insignificant contributions to the scatter in fatigue life compared to the dynamic loading and component geometries.

Keywords: Fatigue life prediction, Probabilistic Methods, Aluminum 6061-T6.

1. Introduction

The topic of structural fatigue is concerned with the mechanical behavior of structures, which are subject to repeated load applications. Fatigue failure is a remarkably complex physical process, which is directed by a several number of parameters such as loading and boundary conditions and repeated stresses over time. Most fatigue failures caused by cyclic loads significantly below the loads that would result in yielding of the material [1]. Cyclic nature of the load that leads to microscopic material flaws to propagate into a macroscopic crack. Due to constant repeated loads, the crack grows to a critical size that results in structural or pressure boundary failure of the component [2]. They occur, in

many cases, without any plastic deformation, which means there is usually no visible warning of that a failure is about to happen.

In the beginning of the material's lifetime, the material experiences only inner defects (micro cracks, voids,..., etc.). Because of the effect of the cyclic loading, the small defects tend to increase in size and shape that effects the material, reducing its stiffness. Then, the defects become small cracks and the failure progression develops into its second stage of small crack growth [3-5]. With a reasonably high number of cycles, these growing small cracks become large cracks, which demonstrates the fracture behavior. Figure 1 summarizes the general understanding of the fracture mechanism. In general, there are four phases of crack growth [6].

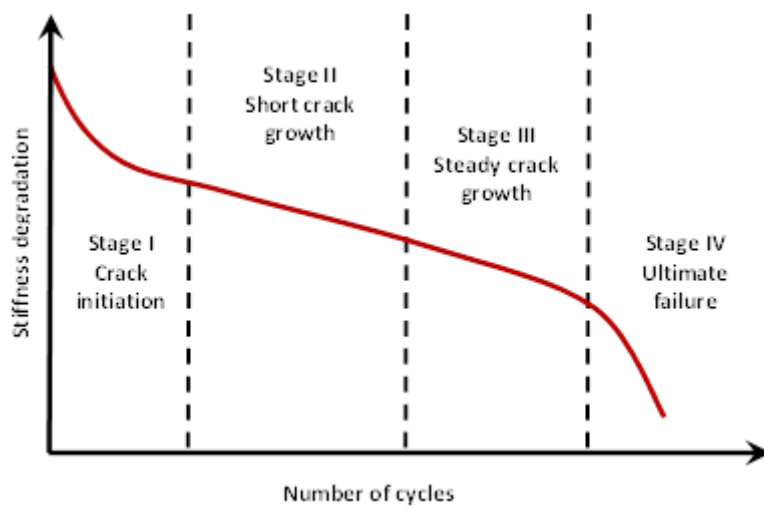


Figure 1. Stages of crack nucleation and growth

Many of the reported strength data are mean values. There is significant scatter in numerous tests of the same material under the same experimental conditions. Haugen and Wirsching described that the standard deviations of endurance strengths of steels rarely exceed 8% of their mean values [7]. The best approach, they suggested, was to develop a set of material-property data for a

particular loading and boundary conditions to a design. Since this method is usually excessively expensive in both time and money, the engineer relies on published material-property data. Some of the tested material samples failed at stresses lower than the mean value, which leads to reduce design strength.

The Stress-Life (S-N), The Strain-Life (ϵ -N), and Linear Elastic Fracture Mechanics (LEFM) are the main models that often used for high-cycle fatigue applications. The optimal approach of fatigue-failure testing for machine-design purposes can be determined by the type of machinery being designed on and its intended use. The large class of rotating machinery, stationary or mobile, is well served by the S-N approach because the required lives are usually in the High-Cycle Fatigue (HCF) range that the other two models work best in Low-Cycle Fatigue (LCF) regimes are more focused on the mechanism of crack initiation or existing crack [8].

Several researchers have examined probabilistic characteristics of fatigue, including inconsistency in S-N data [4,9-16]. Traditionally, the uncertainties in the data have been compensated by applying experience-based safety factors or margins to the calculated maximum load to the design analysis of essential components, leading the probability of failure to a satisfactory level. Different from deterministic evaluations, probabilistic approaches introduce the input variables as distributions and predict distributions of performance [16-19].

Stress-Life predictions using probabilistic fracture mechanics have commonly used inherently time-intensive Monte Carlo simulations [20-22], and simplified fracture mechanics models. In some cases, complicated mathematical manipulations and formulations are required. For that reason, the current work was to explore other probabilistic life prediction approaches and employ them as an alternative to Monte Carlo method to calculate fatigue reliability of Aluminum 6061-T6 sheets. A class of probabilistic approaches based on the most probable point (MPP) are becoming regularly used as a means of reducing the number of performance function calculations from that of exhaustive Monte Carlo

simulation. Even though many variations have been anticipated, the best-known and most widely used MPP-based methods include the first-order reliability method (FORM) [23], mean value (MV), and advanced mean value (AMV) [24].

2. Method

The objective in this study was to provide quantitative measures of component reliability that represents uncertainties inherent to manufacturing tolerances of and material properties by exploring different probabilistic approaches as alternatives to Monte Carlo simulation.

Deterministic model

As previously discussed, components were subjected to a large number of cycles of the applied stress that failed under stress values well below the ultimate strength and often even below yield strength. It has been found that the cycles to failure, N , versus the stress value, S , follow a straight line when plotted on log-log paper. The experimental data used in this study was an original collection of articles by Birnbaum and Saunders [25]. In particular, the study utilized the real data set used in [25-30] and a scaled version of the data set was given by Owen and Padgett [31].

Experimental method

The fatigue-testing machine developed by the Instrument Development Unit of the Physical Research Staff, Boeing Aircraft Company, which designed for metal-coupons to undergo repeated alternating stresses & strains [25, 31]. The coupon (rectangular strips) cut from aluminum sheeting was clamped at the ends and deflected in the center to either side from the position of equilibrium; the frequency of oscillation and stress amplitude was controlled. The amplitude was set either to constant (periodic loading) or random with a determined distribution. The nominal maximum stress was calculated from the maximum amplitude. The machine was equipped with a mechanism to clock the number of cycles that caused fatigue failure of the coupons.

Table 1 shows the number of cycles to failure obtained from there stress levels of fatigue testing of Aluminum 6061-T6 sheet (rectangular cut) ASTM (0.5" x 0.125"). The three sets of data were obtained, each under a periodic loading scheme with frequency of 18 cycles/sec.

Table 1. Failure data ($\text{cycles} \times 10^3$) of 304 6061-T6 aluminum coupons tested for fatigue failure under cyclic stress at three different levels of maximum stress per cycle.

Failure data collected at 21 ksi				Failure data collected at 26 ksi				Failure data collected at 31 ksi			
570	1120	1420	1750	233	352	400	445	70	120	134	149
706	1134	1420	1750	258	356	403	445	90	121	134	151
716	1140	1450	1763	268	358	404	452	96	121	134	151
746	1199	1452	1768	276	358	406	456	97	123	134	152
785	1200	1475	1781	290	360	408	456	99	124	136	155
797	1200	1478	1782	310	362	408	460	100	124	136	156
844	1203	1481	1792	312	363	410	464	103	124	137	157
855	1222	1485	1820	315	366	412	466	104	124	138	157
858	1235	1502	1868	318	367	414	468	104	124	138	157
886	1238	1505	1881	321	370	416	470	105	128	138	157
886	1252	1513	1890	321	370	416	470	107	128	139	158
930	1258	1522	1893	329	372	416	473	108	129	139	159
960	1262	1522	1895	335	372	420	474	108	129	141	162
988	1269	1530	1910	336	374	422	476	108	130	141	163
990	1270	1540	1923	338	375	423	476	109	130	142	163
1000	1290	1560	1940	338	376	426	486	109	130	142	164
1010	1293	1567	1945	342	379	428	488	112	131	142	166
1016	1300	1578	2023	342	379	432	489	112	131	142	166
1018	1310	1594	2100	342	380	432	490	113	131	142	168
1020	1313	1602	2130	344	382	433	491	114	131	142	170
1055	1315	1604	2215	349	389	433	503	114	131	144	174
1085	1330	1608	2268	350	389	437	517	114	132	144	196
1102	1355	1630	2440	350	395	438	540	116	132	145	212
1102	1390	1642		351	396	439	560	119	132	146	
1108	1416	1674		351	400	439		120	133	148	
1115	1419	1730		352	400	443		120	134	148	

Figure 2 shows variability in number of cycles to failure for each stress and the three levels of maximum stress for a periodic loading scheme, $\sigma_{a1} = 21$, $\sigma_{a2} = 26$ and $\sigma_{a3} = 31$ ksi. The data were

originally presented & examined in [26] and partially in [25]. The specimen was 6061-T6 aluminum sheeting, which was cut (parallel to the direction of loading) into the rectangular strips. The testing produced data for the three levels of maximum stress with sample sizes of $n_1=101$, $n_2=102$ and $n_3=101$. The means and standard deviations of all sets were summarized in Table 2 and Figure 2. This data set was investigated in the literature assuming that alternating stress σ_a was constant for each stress level, mean stress σ_m was zero. Stress Life relationship represented as $S = aN^b$, where b is the slope of the curve, a is constant, S is stress, and N is Life or number of cycles to failure.

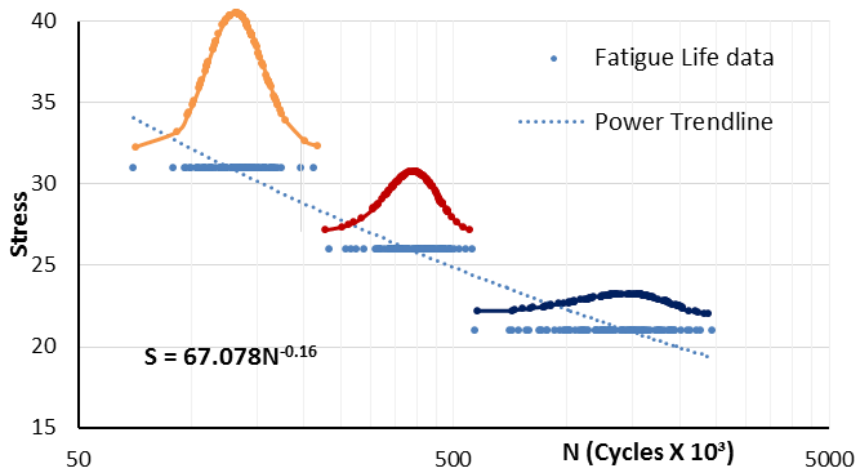


Figure 2. S-N-P curve shows probabilistic nature of fatigue failures at three stress levels.

Table 2. Experimental data distributions

10 ⁵ cycles @	31 ksi	26 ksi	21 ksi
Mean	1.3373	3.9788	14.029
Std.	0.22356	0.62324	3.8653

Probabilistic Model

Although S-N curves are widely used to investigate fatigue failures, they have a number of limitations. For instance, they are

unable to capture the variation of actual components such as geometry, surface finish, and manufacturing defects. Besides that, they ignore variation in actual use conditions of the components in terms of loading, which are usually different from the test conditions and importantly they do not take into account the probabilistic nature of fatigue failures. There has been a better approach that utilizes S-N-P curves that takes probabilistic behavior and those variations into consideration [32]. There are several probabilistic approaches based on different theories and assumptions to study material failure behavior during dynamic loading. Although, selecting one method is desirable, applying several approaches may provide a better realistic prediction on fatigue life in all cases. However, numerous studies used the brute-force Monte Carlo simulation to accomplish that. This study used three different advanced probability methods; MV, AMV and FORM, as alternatives.

Performance Equation

In the current study, the input parameters were modeled as distributions, and perturbed variables for the probabilistic methods were passed to a commercially available probabilistic software package, NESSUS. All variables that contribute in component life as distributions to predict a distribution of performance as shown in the following equation. In the probabilistic analyses, input parameter variability was considered for cyclic loading, component geometry, and slope S-N slope parameters (a and b). Distributions for the input parameters were developed from available literature data for Aluminum 6061-T6 conditions [26, 32].

$$\frac{load_{alternating}(\mu, \sigma)}{Area(with(\mu, \sigma) * thickness(\mu, \sigma))} = stress$$
$$= a(\mu, \sigma) * N^{b(\mu, \sigma)},$$

where μ , σ are the mean values and standard deviations respectively.

Several steps were taken to set up probabilistic model then define the limit state equation. First, goodness of fit showed that fit

normal and lognormal distributions were acceptable at 5% significant level (K-S test). However, lognormal performed better than normal distribution. Second step was to generate distributions with mean and standard deviation of slope parameters. That was achieved by capturing values of the two parameters that were between 3+ standard deviation from the means as visualized in Figure 3. The third step was to estimate 5% lower limit for a and b which was considered the higher risk region. At that point the design curve was obtained and presented in the following equation.

$$N = \left(\frac{S}{67.078} \right)^{\left(\frac{1}{-0.1598} \right)}$$

After obtaining the performance equation, probabilistic models were used to quantify the risk (probability of failure) and compute shortest reliable life for a component using different reliability methods, specifically, a 50k-trial Monte Carlo simulation and the most probable point (MPP) methods; the mean-value (MV), the advanced mean-value (AMV) and the first order reliability method (FORM).

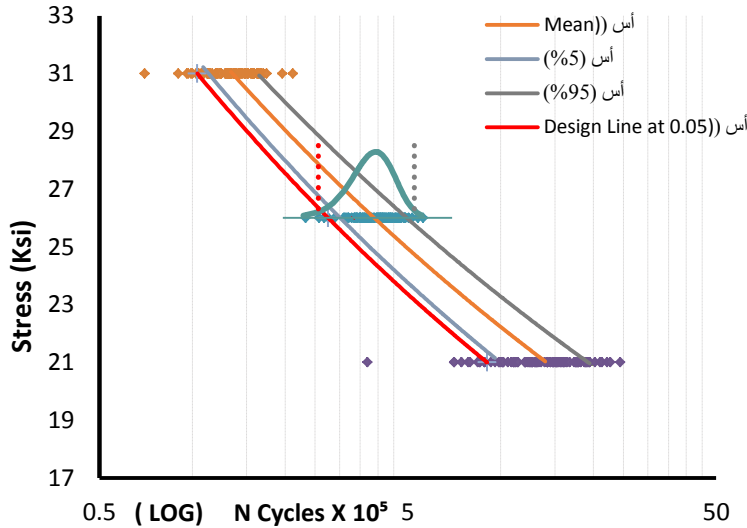


Figure 3. Design curve at 5 % lower bound

Probabilistic Parameters

Table 3 shows inputs of probabilistic study parameters with means and standard deviations. The dynamic load as well as for the geometry variables (width and thickness of the Al-6061-T6 Sheet) were described by normal distributions, while the slope of the S-N curve variables were described by lognormal distributions. The load was assumed in the study that gave 26 ksi stress for the purpose of comparing results with experimental data at the same stress level. In the field, however, they depend upon the use conditions with observed variations. The problem statement with probabilistic inputs were defined in NESSUS.

$$Life = \left(\frac{S}{a} \right)^{\left(\frac{1}{b} \right)}$$

$$S = \frac{F}{Area}$$

$$Area = t * w$$

Table 3. Input distributions

Variables		Mean	Standard deviation
Geometry (normal-distribution)	width	0.5"	+ 0.01"
	thickness	0.125"	+ 0.005"
Load (normal-distribution)	force	1.75	0.1
Slope (lognormal distribution)	a	67.078	0.22
	b	-0.1598	0.0018

Reliability Methods

Monte Carlo simulations and advanced probabilistic methods performed in the current study are well described in the literature [23,33,35] and summarized as follows.

Mean value (MV): Based on approximating MPP by perturbing variables near the mean. Number of trials = 1 + (number of variables).

Advanced mean value (AMV): MV + additional evaluation at the MPP.

Number of trials = 1 + (number of variables) + (number of p levels)

First Order Reliability Method (FORM): various algorithms based on first order approximation of the performance function. Number of trials is dependent on convergence.

Monte Carlo Simulation: Randomly generates parameter values from their distributions.

3. Results and Discussion

Two quantitative measures were employed in the study; comparative analysis among probabilistic methods used in the study and identifying which input parameters are most critical to the performance distribution. The results showed slight differences among reliability methods. Life distribution and sensitivity factors for dominant variables were also obtained.

Fatigue Life Distribution

The model predicted the distribution of life for Al 6061-T6 sheet under axial fatigue stress. The probability of survival or reliability was based on the performance function and represents the probability of failure (POF=0.1) of the number of cycles that components can survive under 26 ksi dynamic loading. Based on the summarized results in Table 4 and cumulative density functions (CDFs) presented in Figure 4. The predicted distributions of life-to-failure showed reasonable agreement with the experimental data over the entire CDF, with accurate predictions of the mean. Moreover, the AMV and FORM methods gave very similar results to the Monte Carlo simulations in less than a second of computation time required only a small fraction of the iterations. The MV predictions varied noticeably from the experimental data for probabilities away from 0.50; this result is somewhat expected as the MV method assumes linear behavior which causes inaccuracies in non-linear systems [24]. To evaluate the accuracy of the different probabilistic methods, their predictions of fatigue life for a fixed probability of failure, POF, of 0.1 were compared in Table 4. This low probability was selected based on conservative design assumptions that require the shortest fatigue lives be considered in design to ensure a higher reliability.

Table 4. Predicted life at POF = 0.1 at 26 ksi stress level

Method		Life (cycles) @ POF = 0.1	Number of trials	Time
Probabilistic	MV	169,595.4 <i>shortest</i>	6	0.32
	AMV	209,007.923	4	0.41
	FORM	207,791.7	5	0.33
	Monte Carlo	208,295.5	50,000	3 minutes
Experimental	Data ($\mu \pm \sigma$)	397,545.0 $\pm$ 62541	102 sample points	

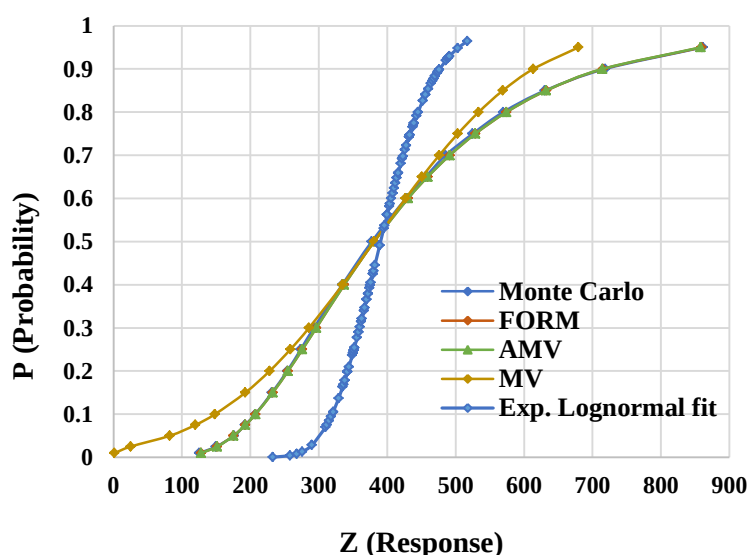


Figure 4. Cumulative distribution functions of performance function from Monte Carlo, FORM, AMV, and MV probabilistic analyses versus Experimental data cumulative distribution function.

Sensitivity Analysis

Sensitivity analysis indicates the effect of each individual parameter on the reliability function. In other words, it illustrates which input parameters that were, during the computation process, the most contributing factors in the component life. Additionally, random variables with negligible contributions can be treated as deterministic variables to reduce the computation time. Figure 5 shows that life to failure of a component was most sensitive to the

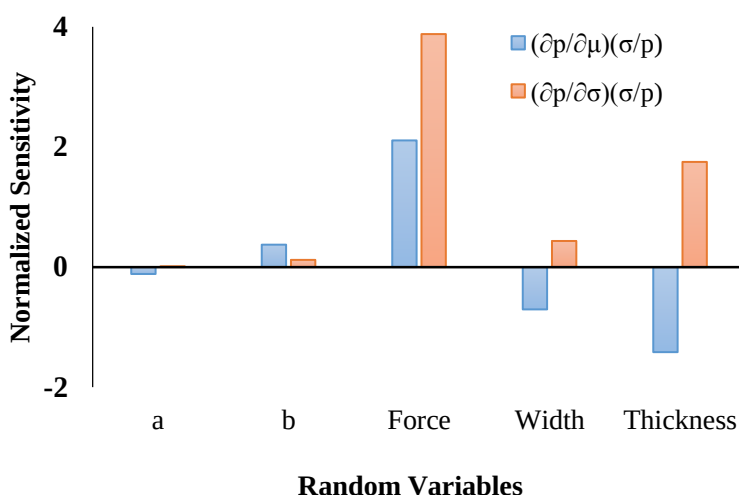
amount of variation in loading. In addition, cross-section area plays an important role in the outcomes. However, slope of the curve parameters were the least contributing factor, subsequently when they modeled as deterministic, there was not significant change in the outcomes

Figure 5. Sensitivity factors for fatigue life predictions (obtained from the AMV method results). The sensitivity factors are normalized with respect to the probability level ($p = 0.01$) and standard deviations of the respective parameters.

4. Conclusion

In this study, it has been seen how principles of mechanical design and reliability can be combined effectively to obtain accurate reliability predictions and to design reliability directly into components that were subject to fatigue failure. While Monte Carlo analysis is often considered the gold standard, its accuracy depends on the number of trials and the sampling error [33]. Furthermore, the study showed that with less computational time, there is a very good agreement for AMV and FORM with Monte

Sensitivity factors for fatigue life predictions



Carlo. Among the other methods used in the study, however, MV was less accurate in tail regions that indicates poor life prediction,

which is a known limitation of the linear method when applied to non-linear models [24]. Besides that, the model best performed in short life modeling. Additionally, the AMV and FORM methods quantified sensitivity factors that emphasizing the most significant variables in probabilistic modeling. The results showed that the S-N slope parameters made negligible contributions to the scatter in fatigue life and therefore could be modeled as deterministic. However, dynamic loading and component geometries were the chief variables.

References

- [1] Antaki, George, and Ramiz Gilada. Nuclear Power Plant Safety and Mechanical Integrity: Design and Operability of Mechanical Systems, Equipment and Supporting Structures. Butterworth-Heinemann, 2014.
- [2] Shankar, Vani, M. Valsan, K. Bhanu Sankara Rao, R. Kannan, S. L. Mannan, and S. D. Pathak. "Low cycle fatigue behavior and microstructural evolution of modified 9Cr–1Mo ferritic steel." *Materials Science and Engineering: A* 437, no. 2 (2006): 413-422.
- [3] DeBartolo, E. A., and B. M. Hillberry. "A model of initial flaw sizes in aluminum alloys." *International Journal of Fatigue* 23 (2001): 79-86.
- [4] Virkler, Dennis Andrew, Brnm Hillberry, and P. K. Goel. "The statistical nature of fatigue crack propagation." (1979): 148-153.
- [5] Wang, G. S. "Intrinsic statistical characteristics of fatigue crack growth rate." *Engineering fracture mechanics* 51, no. 5 (1995): 787-803.
- [6] Gao, Xiaofeng, Georg Koval, and Cyrille Chazallon. "A discrete element model for damage and fracture of geomaterials under fatigue loading." In *EPJ Web of Conferences*, vol. 140, p. 12018. EDP Sciences, 2017.

- [7] E. B. Haugen and P. H. Wirsching, "Probabilistic Design." Machine Design, vol. 47, pp. 10-14, 1975
- [8] Norton, Robert L. "Machine design: an integrated approach." Pearson, 2013
- [9] Castillo, Enrique, A. Fernández-Canteli, and Ali S. Hadi. "On fitting a fatigue model to data." International Journal of Fatigue 21, no. 1 (1999): 97-106.
- [10] Virkler, Dennis Andrew, Brnm Hillberry, and P. K. Goel. "The statistical nature of fatigue crack propagation." (1979): 148-153.
- [11] Wang, G. S. "Intrinsic statistical characteristics of fatigue crack growth rate." Engineering fracture mechanics 51, no. 5 (1995): 787-803.
- [12] Lin, Y. K., and J. N. Yang. "A stochastic theory of fatigue crack propagation." AIAA journal 23, no. 1 (1985): 117-124.
- [13] Wu, W. F., and C. C. Ni. "A study of stochastic fatigue crack growth modeling through experimental data." Probabilistic Engineering Mechanics 18, no. 2 (2003): 107-118.
- [14] Wu, W. F., and C. C. Ni. "Probabilistic models of fatigue crack propagation and their experimental verification." Probabilistic Engineering Mechanics 19, no. 3 (2004): 247-257.
- [15] Jha, S. K., M. J. Caton, and J. M. Larsen. "A new paradigm of fatigue variability behavior and implications for life prediction." Materials Science and Engineering: A 468 (2007): 23-32.
- [16] Enright, Michael P., R. Craig McClung, and Luc Huyse. "A probabilistic framework for risk prediction of gas turbine engine components with inherent or induced material anomalies." In Turbo Expo: Power for Land, Sea, and Air, vol. 47276, pp. 505-512. 2005.

- [17] Liu, Yongming, and Sankaran Mahadevan. "Probabilistic fatigue life prediction using an equivalent initial flaw size distribution." *International Journal of Fatigue* 31, no. 3 (2009): 476-487.
- [18] Grell, William A., and Peter J. Laz. "Probabilistic fatigue life prediction using AFGROW and accounting for material variability." *International Journal of Fatigue* 32, no. 7 (2010): 1042-1049.
- [19] Luo, Jiaming, and Paul Bowen. "A probabilistic methodology for fatigue life prediction." *Acta materialia* 51, no. 12 (2003): 3537-3550.
- [20] Ebbeler, Donald, L. Newlin, and Mircea Grigoriu. "Alternative computational approaches for probabilistic fatigue analysis." In *36th Structures, Structural Dynamics and Materials Conference*, p. 1359. 1995.
- [21] Sain, Trisha, and JM Chandra Kishen. "Probabilistic assessment of fatigue crack growth in concrete." *International Journal of Fatigue* 30, no. 12 (2008): 2156-2164.
- [22] Wu, Y. T., M. P. Enright, R. C. McClung, H. Millwater, and G. R. Leverant. "Probabilistic methods for design assessment of reliability with inspection (DARWINTM)." In *Proceedings, 41st Structures, Structural Dynamics, and Materials Conference*, pp. 3-6. 2000.
- [23] Thacker, B., H. Millwater, and S. Harren. "Computational methods for structural load and resistance modeling." In *32nd Structures, Structural Dynamics, and Materials Conference*, p. 918. 1991..
- [24] Wu, Y-T., H. R. Millwater, and T. A. Cruse. "Advanced probabilistic structural analysis method for implicit performance functions." *AIAA journal* 28, no. 9 (1990): 1663-1669.

- [25] Birnbaum, Z. W., and Sam C. Saunders. "A statistical model for life-length of materials." Journal of the American Statistical Association 53, no. 281 (1958): 151-160.
- [26] Birnbaum, Zygmunt W., and Sam C. Saunders. "Estimation for a family of life distributions with applications to fatigue." Journal of applied probability 6, no. 2 (1969): 328-347.
- [27] Gómez, Héctor W., Juan F. Olivares-Pacheco, and Heleno Bolfarine. "An extension of the generalized Birnbaum–Saunders distribution." Statistics & Probability Letters 79, no. 3 (2009): 331-338.
- [28] Leiva, Víctor, Edgardo Rojas, Manuel Galea, and Antonio Sanhueza. "Diagnostics in Birnbaum–Saunders accelerated life models with an application to fatigue data." Applied Stochastic Models in Business and Industry 30, no. 2 (2014): 115-131.
- [29] Collins, J., 1981. Failure of Materials in Mechanical Design. Wiley, New York.
- [30] Dupuis, Debbie J., and Joanna E. Mills. "Robust estimation of the Birnbaum-Saunders distribution." IEEE Transactions on Reliability 47, no. 1 (1998): 88-95.
- [31] Owen, W. Jason, and William J. Padgett. "A Birnbaum-Saunders accelerated life model." IEEE Transactions on Reliability 49, no. 2 (2000): 224-229.
- [32] Kececioglu, Dimitri. Robust engineering design-by-reliability with emphasis on mechanical components & structural reliability. Vol. 1. DEStech Publications, Inc, 2003.
- [33] Halder, Achintya, and Sankaran Mahadevan. Reliability assessment using stochastic finite element analysis. John Wiley & Sons, 2000.
- [34] Melchers, Robert E., and André T. Beck. Structural reliability analysis and prediction. John wiley & sons, 2018.